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ACCURACY MODEL OF ROTARY INDEXING TABLE

The subject of this paper is an accuracy model of rotary indexing table based on Hirth coupling. Positioning accuracy of the table depends on the numerous factors, out of which pitch error plays the most important role. Theorem of order statistics and its application in modeling positioning accuracy is described. Based on the derived model, probability density function of maximal pitch values are presented in the paper. Linking maximal pitch error to the positioning accuracy is explained and the average accuracy errors of the rotary indexing table, depending on the number of gear teeth is shown. In order to compare the given results, Hirth couplings were manufactured and their pitch error was measured. It has been stated that positioning accuracy rises together with the total number of teeth.

Key words: accuracy model, order statistics, rotary indexing tables, Hirth coupling

1. INTRODUCTION

There is a growing tendency in the development of modern machine tools including rotary axis to improve their performance in terms of dynamics, accuracy and cost. This phenomenon triggers the advancements in control systems, design and materials in terms of thermal stability, damping and rigidity. In addition, new and more precise diagnostics systems are introduced to follow the increasing precision. Kinematic structure of machine tools changes from classic serial to closed parallel in order to achieve high speed and acceleration while maintaining the required machining accuracy [9].

The positioning accuracy of rotary axis depends directly on the applied table. The method of analytical examinations of the influence of geometric errors in linear guideway on joint kinematic errors, taking into account the deformations of the table, was described by Majda [5]. The subject of this paper is a rotary indexing table where positioning is done by the pair of Hirth face gears. The positioning

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error is determined directly by the pitch error of each gear. The factors affecting the precision of teeth machining can be divided into two categories:

- deterministic (e.g. tool positioning error, positioning error of a table on which the cut coupling is fixed),

Applying Hirth coupling (see Figure 1) to the table stems from its ability to transmit relatively high torque (up to hundreds of kN·m) and its theoretical contact ratio equal to the number of teeth. Manufacturers claim their other advantages:

- high accuracy (up to a few seconds) and repeatability,
- self-centering,
- low axial and radial run-out,
- high durability.

There are numerous articles related on modeling the accuracy of rotary indexing tables based on the kinematic errors. Hong et al. presented the influence of position-dependent geometric errors of rotary tables on a machining test of cone frustrum [3]. Ibaraki et al. presented the method of constructing error map of rotary axes by means of on-the-machine measurement machine measurement of test pieces by using a contact-type touch-trigger probe installed on the machine’s spindle [4].

All the aforementioned papers do not consider the impact of the pitch error on the overall accuracy of indexing tables. Staniek suggested that there is a connection between the contact ratio and the location error of gear train [9]. It was shown that the relative location error \( k_i \) could be described by the formula combining the contact ratio and the number of teeth [9]:

\[
k_A = 1 - \frac{\log \varepsilon}{\log z}
\]  

(1)

It can be noticed that for the gears with contact ratio equal to the number of teeth, e.g. Hirth coupling, theoretical relative error should be zero no matter the number of the teeth. That hypothesis was discussed in this paper.

There are numerous standards related to positioning accuracy of rotary axis. In ISO230-7 [6], location errors of a rotary axis constitute position and orientation errors of the axis average line of a rotary axis, i.e. the straight line representing the mean location and orientation of its axis of rotation. Location errors are one of the most essential error determinants in the complex four- and five-axis kinematics. ISO 10791-1 [7] contains quasi-static or no-load tests focusing on calibrating location errors of rotary axes. In the annex A, it considers 45° split indexable heads,
with mechanical indexing of the different angular positions of the two bodies (e.g. Hirth couplings). For calibrating axial location error ball bar measurements are applied [1, 10]. The R-test [2, 11] extends the measurements to three-dimensional space. Dynamic interpolation tests using the ball bar or the R-test are the subject of ISO/DIS10791-6:2012 [8].

2. STATISTICAL MODEL

The accuracy model was derived with the assumption that the pitch error is a random variable following certain probability distribution. Such approach allows to clearly describe probability density function and its basic parameters: mean, standard deviation and skewness. However, it does not cover deterministic factors influencing the geometrical accuracy of teeth e.g. clearances in the machine tool mechanisms, tool wear, geometrical errors of the workpiece. Based on the number of teeth, statistical model was devised taking into account:

- pitch error as an individual teeth is independent random variable,
- probability density function describing every teeth are the same,
- teeth are ideally rigid,
- teeth of one gear are free of geometrical errors.

According to the aforementioned assumptions, in every pair of contacting teeth one of them is characterized by the pitch error, the other is ideal. This situation can be considered as if there is a random sample of $z$-elements.

Let $Y_i$ (continuous random variable) describes the pitch value of contacting $i$-tooth. All the variables $Y_i$ are characterized by the same normal distribution:

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right)$$

(2)

where: $\mu$ – pitch nominal value,
$\sigma$ – standard deviation of $Y_i$ variable.

$Y_i$ variables constitutes $z$-dimensional random vector:

$$Y_i = (Y_1, Y_2, ..., Y_z)$$

(3)

Observed pitch values for $z$ teeth constitute random sample:

$$y_i = (y_{i1}, y_{i2}, ..., y_{iz})$$

(4)

Standardized random variables $X_i$ can be denoted as:
\[ X_i = \frac{Y_i - \mu_{\text{nom}}}{\sigma} \]  

(5)

\( X \) variables follow normal distribution \( N(0,1) \), characterized by the probability density function

\[
f(x) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right)\]

(6)

The standardized random vector

\[ X_i = (X_1, X_2, \ldots, X_z) \]

(7)

corresponds to vector (3) and standardized random sample

\[ x_i = (x_1, x_2, \ldots, x_z) \]

to the sample (4).

Maximal values of the pitch error play the most important role in terms of positioning error of the rotary indexing table. According to the statics, it requires three points to completely constrain the rigid body. Simplifying the gear movement to only one rotation, the positioning takes place at one point. The most significant in the model is to determine maximal pitch value \( p_{\text{max}} \) depending on the number of teeth \( z \).

If elements \( x_i \) of every sample (8) are arranged in decreasing order, new sample is derived:

\[ (x_{r_1}, x_{r_2}, \ldots, x_{r_z}) \]

(9)

following the inequality

\[ (x_{r_1} \leq x_{r_2} \leq \ldots \leq x_{r_z}) \]

(10)

Considering \( N \) random samples (8) and arranging them in decreasing order, new order statistics \( \xi_k^{(z)} \) for \( k = 1, 2, \ldots, z \) are obtained. For example, \( \xi_1^{(z)} \) consists of the lowest values of every sample.
According to the introduced definition of positioning error, the most essential is the maximal value of pitch in every random sample. Calculating the positioning error requires determining maximal values of the samples (8) e.i. distribution of the statistics described by the cumulative distribution function $\Phi_{\xi_z}$:

$$\Phi(w) = P(\xi_z < w)$$  \hspace{1cm} (11)

where $P$ is the probability of the event $\xi_z < w$ meaning that during $z$ observations the maximal pitch value is lower than certain value $w$. If the maximal value of pitch meets inequality (11) then all the observations from the sample takes lower values than $w$. For the given $w$, using Bernoulli scheme, probability $P$ takes the following form:

$$\Phi(w) = P(\xi_z < w) = \left( \frac{z}{z} \right)^z p^z$$  \hspace{1cm} (12)

where $p$ is the probability of $X_i$ taking values lower than $w$.

Following the assumption of standard distribution of $X_i$, probability $p$ can be denoted as:

$$p = P(X_i < w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} \exp \left( -\frac{x^2}{2} \right) dx$$  \hspace{1cm} (13)

Considering in the formula (11) equation (12) of the probability $p$, probability density function is derived:

$$\phi(w) = \frac{z}{(\sqrt{2\pi})^z} \left[ \int_{-\infty}^{w} \exp \left( -\frac{x^2}{2} \right) dx \right]^{z-1} \exp \left( -\frac{w^2}{2} \right)$$  \hspace{1cm} (14)

Probability density functions obtained for various number of teeth $z$ are depicted in Figure 2.

For the given probability distribution function mean value can be calculated referring to relative mean value of maximal pitch error depending on the number of teeth of Hirth coupling. Expected value can be denoted as:

$$\delta_p = \int_{-\infty}^{\infty} w \cdot \phi(w) dw =$$

$$= \int_{-\infty}^{\infty} w \cdot \frac{z}{(\sqrt{2\pi})^z} \left[ \int_{-\infty}^{w} \exp \left( -\frac{x^2}{2} \right) dx \right]^{z-1} \exp \left( -\frac{w^2}{2} \right) dw$$  \hspace{1cm} (15)
Mean value of maximal pitch error is shown in Figure 3. It can be noticed that relative pitch error rises with the increase of number of teeth. It can be explained by the fact that it is more likely to achieve higher value of pitch error (draw a tooth with higher value of pitch error) if higher number of teeth is cut (if more samples are drawn).

![Figure 3. Mean relative maximal pitch error value depending on the number of teeth of Hirth coupling](image)

Assuming that the nominal pitch value depends on the number of gear teeth:

\[ p_{\text{nom}} = \frac{2\pi}{z} \]

(16)

the absolute positioning error value can be calculated by the formula:

\[ \bar{\delta}_d = \delta_p \cdot p_{\text{nom}} \]

(17)

![Figure 4. Mean absolute pitch error value depending on the number of teeth of Hirth coupling](image)
The absolute positioning error depending on the number of teeth is depicted in Figure 4. It can be seen that with an increase of number of teeth the positioning error takes lower values despite the fact that the relative maximal pitch error value increases, as the nominal pitch value decreases and has more impact on the absolute positioning error value. The overview of the relation (17) comparing to the relation (15) is shown in Figure 5.

3. EXPERIMENTAL RESULTS

3.1. Pitch errors

In order to verify the devised model three sets of Hirth couplings were cut. Two of them, having 150 teeth, were manufactured on DMU 60T monoBLOCK five axis machining centre by DMG. The other set – 144 teeth – was cut on FYN 50ND CNC milling machine by Jafo Jarocin. The face teeth were formed by indexing movements of rotary indexing CNC table. The sets were measured on DEA Global Image 775 coordinate measurement machine fitted with RENISHAW SP25M scanning head. CMM was located in air-conditioned room, where the following parameters were noted:

Figure 6. Measuring face gear geometry around three diameters on CMM
– air temperature – \( t = 20^\circ \text{C} \),
– relative humidity – \( f = 40 \% \).

No changes of those conditions were recorded during the gear measurements.

The results of the measurements was a point cloud referring to the shape of every tooth around three diameters: mean, mean \(+ 10 \text{ mm} \) and mean \( - 10 \text{ mm} \) (see Figure 6). Based on the measured data pitch error values of every teeth were calculated using CATIA. The example results of pitch errors are depicted in Figure 7.

![pitch error graph](image-url)

Figure 7. Absolute pitch errors of the face gear of 144 teeth, formed on FYN 50ND CNC milling.

### 3.2. Model verification

The model described in chapter 2 based on the assumption that pitch error values follow normal distribution. Pitch error values of every gear were tested using Kolmogorov-Smirnov [11] test with a view to determining their statistical distribution. The test is used for verifying the hypothesis stating that two distributions are the same or that one empiric distribution is different from an assumed theoretical distribution considering a certain finite number of observations.

In the test, the null hypothesis is stated that the empirical distribution of pitch error values is similar to the normal distribution:

\[
H_0: F_1(x) = F_2(x) \quad (18)
\]

versus the alternative one:

\[
H_1: F_1(x) \neq F_2(x) \quad (19)
\]

where: \( F_1(x) \) – empirical distribution of pitch error derived from measurements,
\( F_2(x) \) – normal distribution \( N(m, \sigma) \), which was assumed in chapter 2.
Verification of the null hypothesis (18) requires to calculate the empirical value of \( \lambda \) statistics:

\[
\lambda = D \cdot \sqrt{n}
\]

where

\[
D = \sup |F_1(x) - F_2(x)|
\]

If \( \lambda < \lambda_0 \) for the given statistical significance it means that the null hypothesis is accepted. The value of \( \lambda_0 \) are derived from Kolmogorov distribution tables as \( F(\lambda_0) = 1 - \alpha \). For \( \alpha = 0.05 \), Kolmogorov statistics takes the value \( \lambda_{0.05} = 1.35 \). Empirical values of \( \lambda \) statistics as well as test verification results for the given set of face gears are presented in Table 1.

<table>
<thead>
<tr>
<th>Machine</th>
<th>FYN 50ND CNC milling machine</th>
<th>DMU 60T machining centre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>144</td>
<td>150</td>
</tr>
<tr>
<td>Gear number</td>
<td>gear 1–1</td>
<td>gear 2–1</td>
</tr>
<tr>
<td></td>
<td>gear 1–2</td>
<td>gear 2–2</td>
</tr>
<tr>
<td></td>
<td>gear 3–1</td>
<td>gear 3–2</td>
</tr>
<tr>
<td>( D )</td>
<td>0.08</td>
<td>0.09</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>0.96</td>
<td>1.08</td>
</tr>
<tr>
<td>Verification</td>
<td>&lt;1.36</td>
<td>&lt;1.36</td>
</tr>
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<td></td>
<td>&lt;1.36</td>
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<tr>
<td></td>
<td>&gt;1.36</td>
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</tbody>
</table>

Two gears were excluded from further analysis as they did not comply with the assumption of normal distribution of pitch errors, which was described in chapter 2. It might be explained by the fact that the impact of systematic errors was greater than stochastic. In order to check if there is a correlation between the number of teeth and the pitch error, maximal pitch error values were calculated for every gear and tau-Kendall test was performed to determine the monotonicity of relation of maximal pitch error depending on the number of teeth:

\[
f(z) = \delta_{\max}(z)
\]

Tau-Kendall value constitutes a difference between the probability that compared variables would be arranged in the same order for two observations and the probability that they would arranged otherwise. If tau takes the value of 1 or –1, it means that every variable increases or decreases with the increase of the other variable. According to the equation (17), the absolute pitch error decreases together with the number of teeth. Tau-Kendall rank correlation coefficient provides a statistical measure to verify if the empirical values of pitch errors decrease as the numbers of gear teeth rise. Values tau were calculated for maximal pitch terror
values using ANALYZE-IT add-on to MS EXCEL 2007. For the considered series of face gears $\tau$ takes a value of $-0.82$.

The negative tau-Kendall value indicates that there is a negative correlation between maximal pitch error and the number of teeth. Therefore, it can be implied that, considering a stable manufacturing process, face gears of higher number of teeth are more convenient for precise rotary positioning tables due to the expected lower value of absolute pitch error affecting the overall positioning accuracy.

4. SUMMARY

Presented statistical model, assuming normal distribution of pitch error has indicated that the mean maximal pitch error increases with the number of teeth of a face gear. For the cut set of Hirth couplings, maximal pitch error value grows as the number of teeth decreases. Linking maximal pitch error with the positioning accuracy leads to a conclusion that for the same technology (including machine accuracy) the positioning error decreases as the teeth number rises. It can be assumed that for the stable manufacturing accuracy of machine tool it is better to cut face gear of lower nominal pitch in order to achieve higher positioning accuracy. A required accuracy of rotary positioning table based on Hirth coupling can be achieved not only by increasing the accuracy of manufacturing process but also by increasing the number of gear teeth as well.

Furthermore, the number of observations plays the vital role, while measuring the quantities which values depend on the highest values captured (such as relative pitch error or flatness). Together with increasing number of measurements it is more probable to find value which is higher than those already measured.

REFERENCES

MODEL DOKŁADNOŚCI PODZIAŁU
DYSKRETNEGO POZYCJONERA OBROTOWEGO

Streszczenie


Słowa kluczowe: dokładność, statystyka pozycyjna, stoły obrotowe, użębienie Hirtha

Received by Editorial Office 3.07.2013